Chapter 1 Savings Choice

Principles of Economics

week 8 2025

1 Model Introduction

In previous lessons, we reviewed the **theory of consumer choice** and used it to provide a preliminary explanation of how people allocate funds between the present and the future. Similarly, it is easy to identify the supply and demand model for funds (savings). For ease of explanation and proof, we will first consider a **two-period model** of fund allocation—specifically, people's savings choices between the current year and the next—and develop a two-dimensional model for derivation. Subsequently, we will extend this model to **n-dimensional and infinite-dimensional forms.**

2 Savings Model Without Currency

First, let us consider how people would make savings decisions between the present and the future in an economy without money.

Here, let us denote c_1 and c_2 as the consumption levels for the current year and the next year, respectively, y_1 as the current year's income, S as the current year's savings, ρ as the discount rate and r as the real interest rate.

Then we can derive the relationship they satisfy and the corresponding utility function:

$$U(c_1, c_2) = u(c_1) + \frac{u(c_2)}{1+\rho} \tag{1}$$

$$\begin{cases} c_1 + s = y_1 & (2) \\ c_2 = (1+r)s & (3) \end{cases}$$

Combining equations (2) and (3):

$$c_2 = (1+r)(y_1 - c_1) \tag{4}$$

$$c_1 + \frac{1}{1+r}c_2 = y_1 \tag{5}$$

The problem now reduces to maximizing the utility function (1) subject to the constraints given by (2) and (3), by differentiation (1) we got:

$$u'(c_1^*) - \frac{1+r}{1+\rho}u'((1+r)(y_1 - c_1^*)) = 0 \tag{6}$$

Which c_1^*, c_2^*, s^* here represent the values of these variables under the maximum condition. Thus we would obtain that

$$\begin{cases} \dfrac{u'(c_1^*)}{u'(c_2^*)} = \dfrac{1+r}{1+\rho} & (7) \\ c_2^* = (1+r)(y_1 - c_1^*) = (1+r)s^* & (8) \end{cases}$$

(7) describes the state of optimal equilibrium an individual achieves when trading off between current and future consumption. The marginal utility ratio on the left side measures the individual's subjective preference for the present relative to the future, while the right side represents the trade-off between the market return (i.e., the real interest rate) and the discount rate (the cost of waiting).

- If $r < \rho$, it implies that the utility loss from waiting outweighs the return offered by the market, leading the individual to favor current consumption over saving.
- Conversely, if $r > \rho$, it means the market return exceeds the individual's impatience, prompting them to reduce current consumption and increase savings for future consumption.

(8) represents the intertemporal budget constraint, defining how the individual transforms current savings into future consumption, thereby ensuring the feasibility of consumption choices.

Furthermore, (7) can be rearranged to isolate the market-determined real interest rate on one side and the individually subjective discount factor $1 + \rho$ on the other,

$$\frac{u'(c_1^*)}{\frac{u'(c_2^*)}{1+\rho}} = 1 + r \tag{9}$$

which will more intuitively reveal the relationship between individual preference choices and market conditions.

3 Savings Model in a Monetary Economy

The introduction of money fundamentally enriches the intertemporal choice model. An individual's current choices are also influenced by their expectations for the future.

We retain the two-period framework but introduce new variables:

Nominal Variables:
P_1, P_2: The price levels in current and next year, noted that P_2^e describes the price level **expected**.
i: The nominal interest rate.

Y_1: The earnings of the first year evaluated in the money
Real Variables: All variables from the previous model (c_1, c_2, y_1, r) are expressed in real terms (units of consumption).

Similarly the equations and functions, only added the influence of nominal variables:

$$U(c_1, c_2) = u(c_1) + \frac{u(c_2)}{1+\rho} \tag{1}$$

$$\begin{cases} P_1 c_1 + s = Y_1 & (10) \\ P_2^e c_2 = (1+i)s & (11) \end{cases}$$

$$\implies c_1 + \frac{1}{1+i}\frac{P_2^e}{P_1}c_2 = \frac{Y_1}{P_1} = y_1 \tag{12}$$

Here, we define a new variable $\pi^e = P_2/P_1$ as an index measuring the price change from period 1 to period 2. More generally, it is commonly referred to as the **Expected Inflation Rate**.

Thus, (12) can be simplified to

$$c_1 + \frac{1+\pi^e}{1+i}c_2 = y_1 \tag{13}$$

Comparing to (5), it is easily to notice that an equation exists between i, r, π^e, which is called the the **Fisher Equation**:

$$1 + i = (1+r)(1+\pi^e) = 1 + r + \pi^e + r\pi^e = 1 + r + \pi^e + o(1) \tag{14}$$

Under normal circumstances, when both the real interest rate and the expected inflation rate are relatively low, the cross term $r\pi^e$ can be treated as an infinitesimal quantity. This allows the Fisher equation to be approximated as:

$$i = r + \pi^e \tag{15}$$

Note 1. However, in cases of high inflation, high real interest rates, or when precise calculations are required, this cross term cannot be omitted.

The maximizing condition is similar to the previous model:

$$u'(c_1^{**}) - \frac{1+i}{1+\pi^e}\frac{1}{1+\rho}u'(\frac{1+i}{1+\pi^e}(y_1 - c_1^{**})) = 0 \tag{16}$$

$$
\begin{cases}
\dfrac{u'(c_1^{**})}{u'(c_2^{**})} = \dfrac{1+i}{1+\pi^e}\dfrac{1}{1+\rho} & (17)\\
c_2^{**} = \dfrac{1+i}{1+\pi^e}\dfrac{1}{1+\rho}(y_1 - c_1^{**}) = \dfrac{1+i}{1+\pi^e}\dfrac{1}{1+\rho}s^{**} & (18)
\end{cases}
$$

4 Savings Model with 2nd Income

Next, let's consider the introduction of a second period of income and observe how the model will change. The constraint equations for consumption, savings, and income become:

$$
\begin{cases}
P_1 c_1 + s = Y_1 & (19)\\
P_2 c_2 = (1+i)s + Y_2 & (20)
\end{cases}
$$

$$
c_1 + \frac{1}{1+i}\frac{P_2}{P_1}c_2 = \frac{Y_1}{P_1} + \frac{\frac{Y_2}{1+i}}{P_1} \tag{21}
$$

As mentioned above, if we have $1+i = (1+r)(1+\pi)$, where we define $1+\pi = \frac{P_2}{P_1}$, The 2nd income item in real terms will become:

$$
\frac{\frac{Y_2}{1+i}}{P_1} = \frac{\frac{Y_2}{(1+r)(1+\pi)}}{P_1} = \frac{\frac{Y_2}{1+\pi}}{(1+r)P_1} = \frac{Y_2\frac{P_1}{P_2}}{(1+r)P_1} = \frac{y_2}{1+r} \tag{22}
$$

Which means the original equation in real terms will be:

$$
c_1 + \frac{1}{1+r}c_2 = y_1 + \frac{1}{1+r}y_2 \tag{23}
$$

5 An Example of a Housing Rental Market

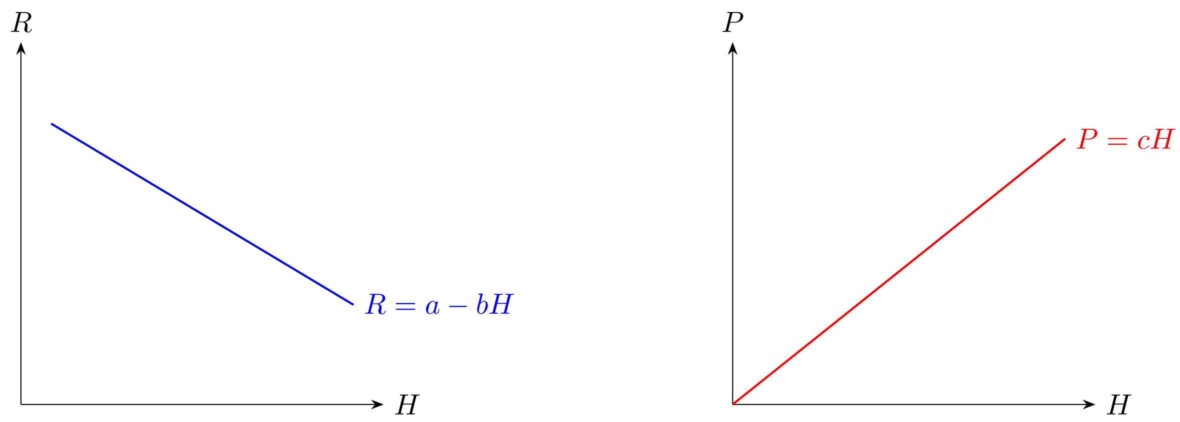
Demand of house renting:

$$
R = a - bH \tag{24}
$$

The will of the householders to supply their house:

$$
P = cH \tag{25}
$$

H : the total built amount of houses
R : the house renting price
P : the marginal cost to build new houses
a, b, c : parameters in the supply curve and demand curve



(a) Rental Demand Curve

(b) Supply Curve

Figure 1: Housing Market: Demand and Supply Relationships

If we define I as the total investment amount, which is equal to the total expenditure on housing construction, we could drop to the equation that

$$I = PH = cH^2 \tag{26}$$

The differential relationship between income I and the total housing stock H is:

$$\frac{dH}{dI} = \frac{1}{2\sqrt{cI}} \tag{27}$$

Define the return rate on investment for housing r as:

$$r = \frac{R}{P} \tag{28}$$

Then the differential relationship between r and I is:

$$
\begin{aligned}
\frac{dr}{dI} &= \frac{d}{dI}\left(\frac{a-b\sqrt{\frac{I}{c}}}{\sqrt{cI}}\right) \\
&= -\frac{b}{2cI} - (a-b\sqrt{\frac{I}{c}})\frac{1}{2\sqrt{c}} \\
&= \frac{1}{\sqrt{cI}}\left(\frac{1}{2\sqrt{cI}}\right)(-b) + \frac{-(a-b\sqrt{\frac{I}{c}})}{cI}\left(\frac{1}{2\sqrt{cI}}\right)c \\
&= \frac{1}{P}\frac{dH}{dI}\frac{dR}{dH} + \frac{-R}{P^2}\frac{dH}{dI}\frac{dP}{dH}
\end{aligned}
$$

Differential Term	**Expression and Source**	**Sign**	**Economic Interpretation**
$\frac{dH}{dI}$	From $I = cH^2 \Rightarrow H = \sqrt{I/c}$	**Positive (+)**	More investment leads to more housing supply
$\frac{dR}{dH}$	From $R = a - bH \Rightarrow \frac{dR}{dH} = -b$	**Negative (-)**	Increased housing supply reduces rental price (law of demand)
$\frac{dP}{dH}$	From $P = cH \Rightarrow \frac{dP}{dH} = c$	**Positive (+)**	Increased housing raises marginal construction costs
$\frac{1}{P}$	Marginal cost $P > 0$	**Positive (+)**	–
$\frac{-R}{P^2}$	Rental price $R > 0$, $P^2 > 0$	**Negative (-)**	–

Combined Effect:

- **First part:** $(1/P) \cdot (dH/dI) \cdot (dR/dH) = (+) \cdot (+) \cdot (-) =$ **Negative**
- **Second part:** $(-R/P^2) \cdot (dH/dI) \cdot (dP/dH) = (-) \cdot (+) \cdot (+) =$ **Negative**

Since both terms are negative, their sum $\frac{dr}{dI}$ must be negative.

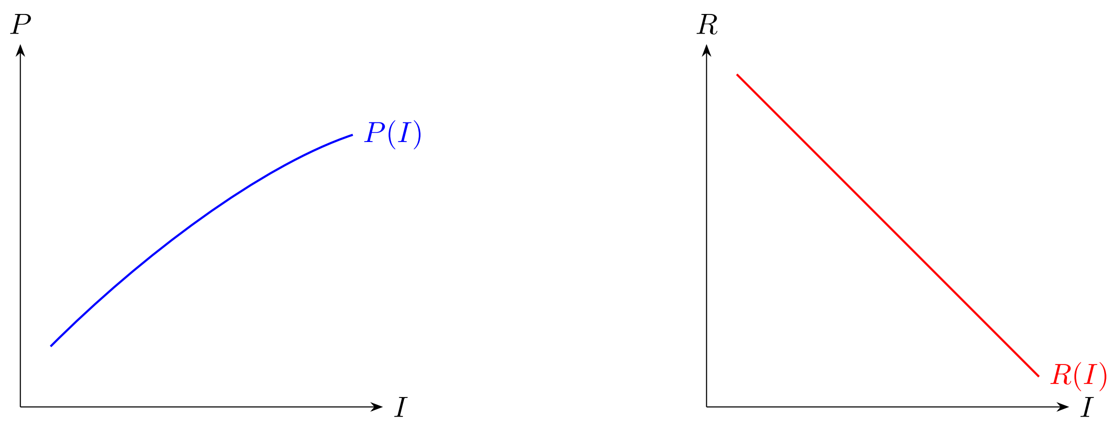
Economic Intuition

This mathematical conclusion has clear economic interpretations:

- **Numerator decreases, denominator increases:**
  - When investment I increases, housing supply H increases

- **Numerator R (rental price):** Due to the downward-sloping demand curve, more housing supply causes rental prices to fall
- **Denominator P (cost):** Due to the upward-sloping supply curve, more housing increases marginal construction costs
- With the numerator decreasing and denominator increasing, their ratio $r = R/P$ naturally decreases

- **Diminishing marginal returns:**
  - In a fixed land and market environment, initial investments yield high returns
  - As investment scale expands, the marginal benefit (reflected in rental price R) decreases while marginal cost (P) increases
  - This causes the overall investment return rate r to decline with increasing investment



(a) Production Function

(b) Demand Curve

Figure 2: Housing Market Relationships with Investment

When we fix the short-term housing supply P, i.e., assume that the housing quantity H remains unchanged in the short term, the housing demand R can be expressed as

$$R = \frac{a - bH}{r} \tag{29}$$